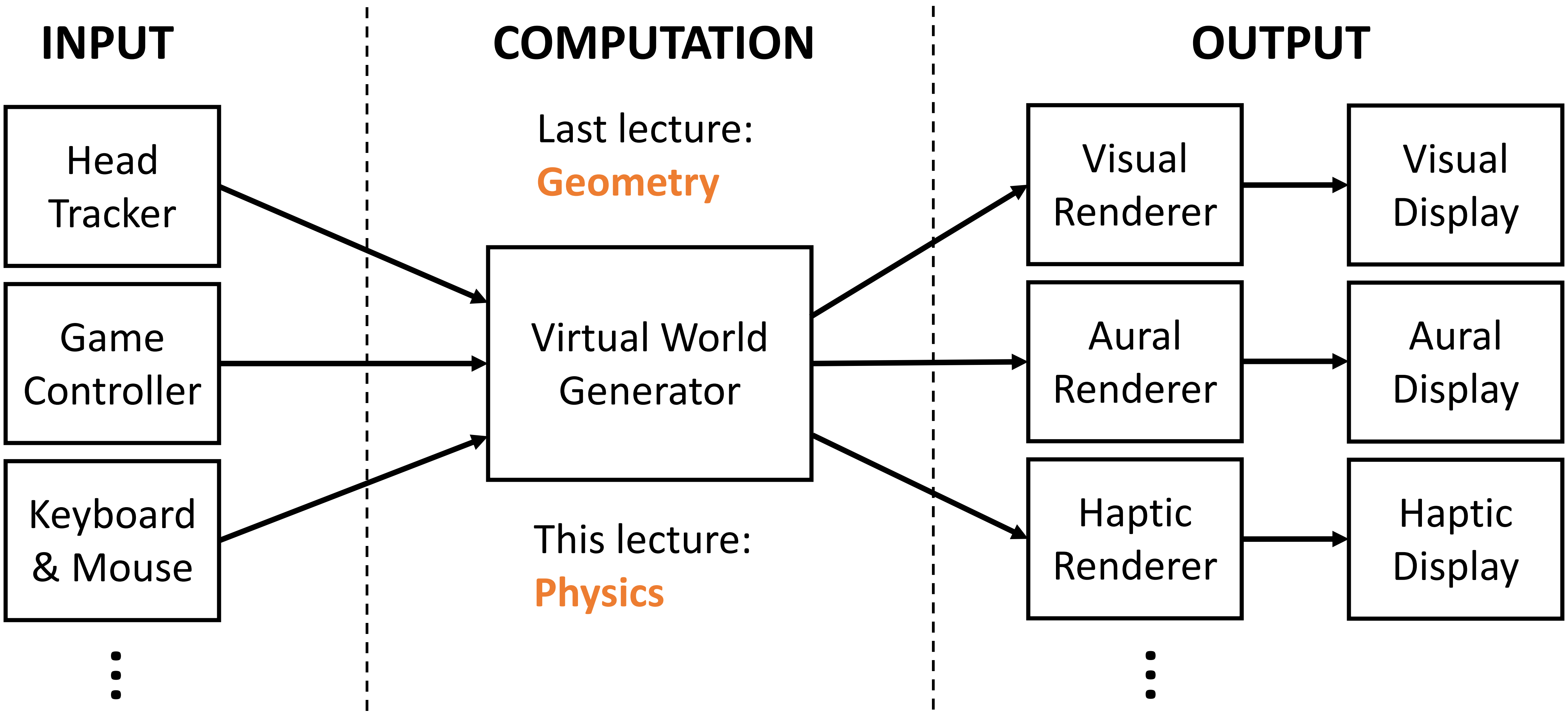
The Physics of Virtual Worlds

CS 6334 Virtual Reality

Professor Yapeng Tian

The University of Texas at Dallas

Review of VR Systems

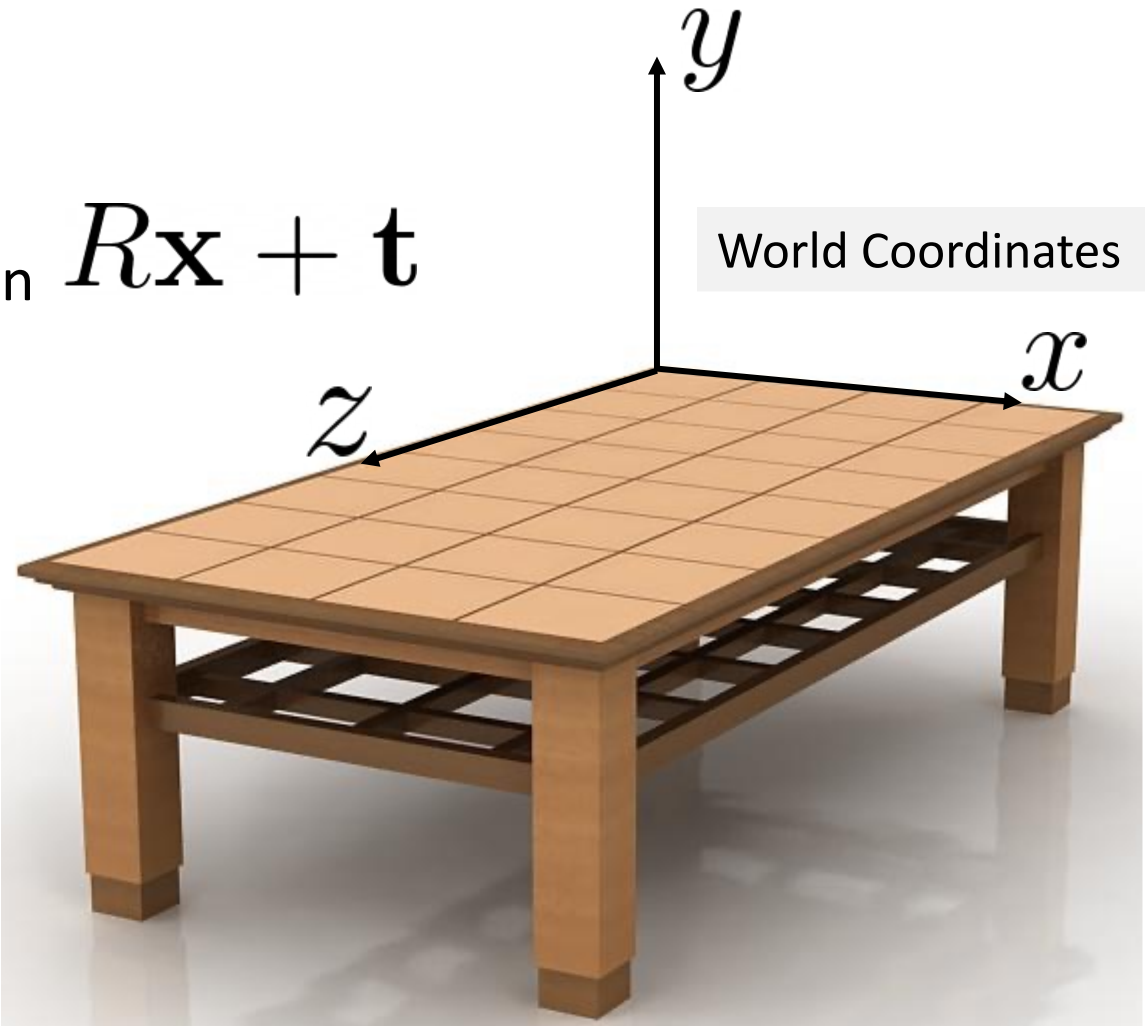


The Geometry of Virtual Worlds

3D Rotation
3D Translation

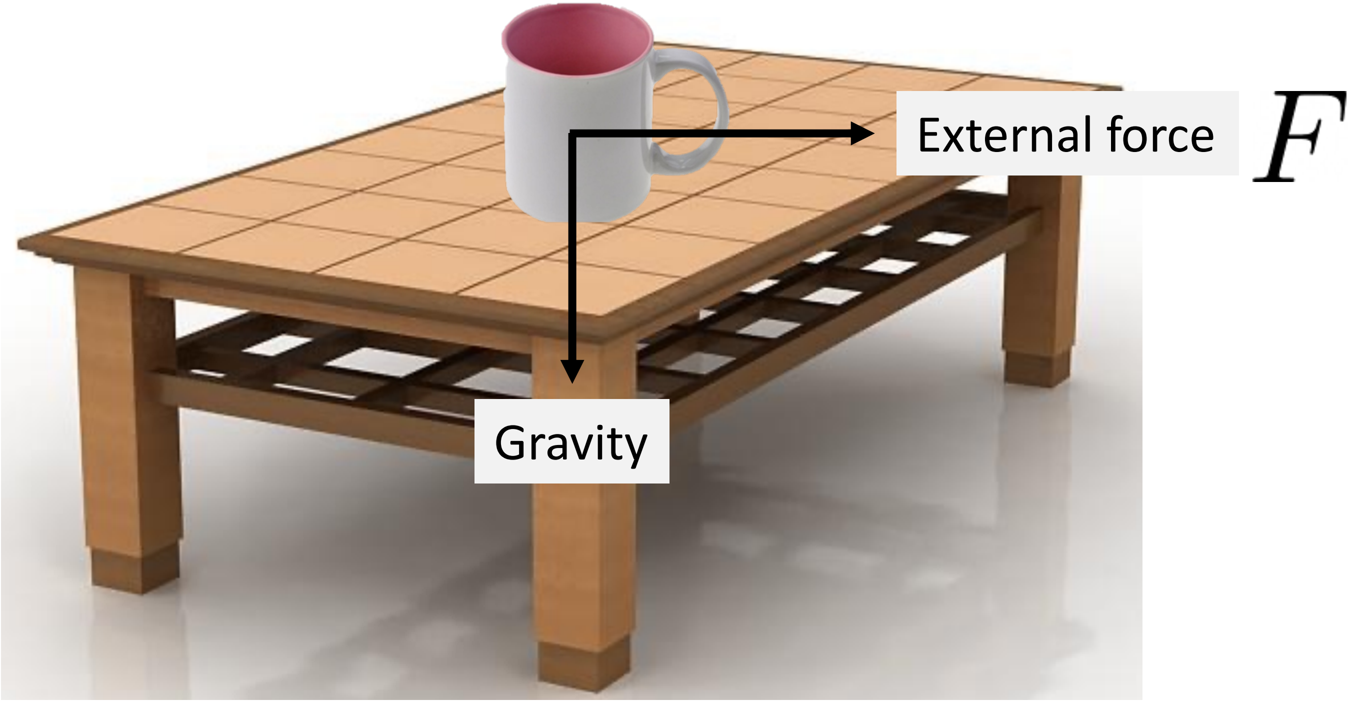
$$R\mathbf{x} + \mathbf{t}$$

World Coordinates



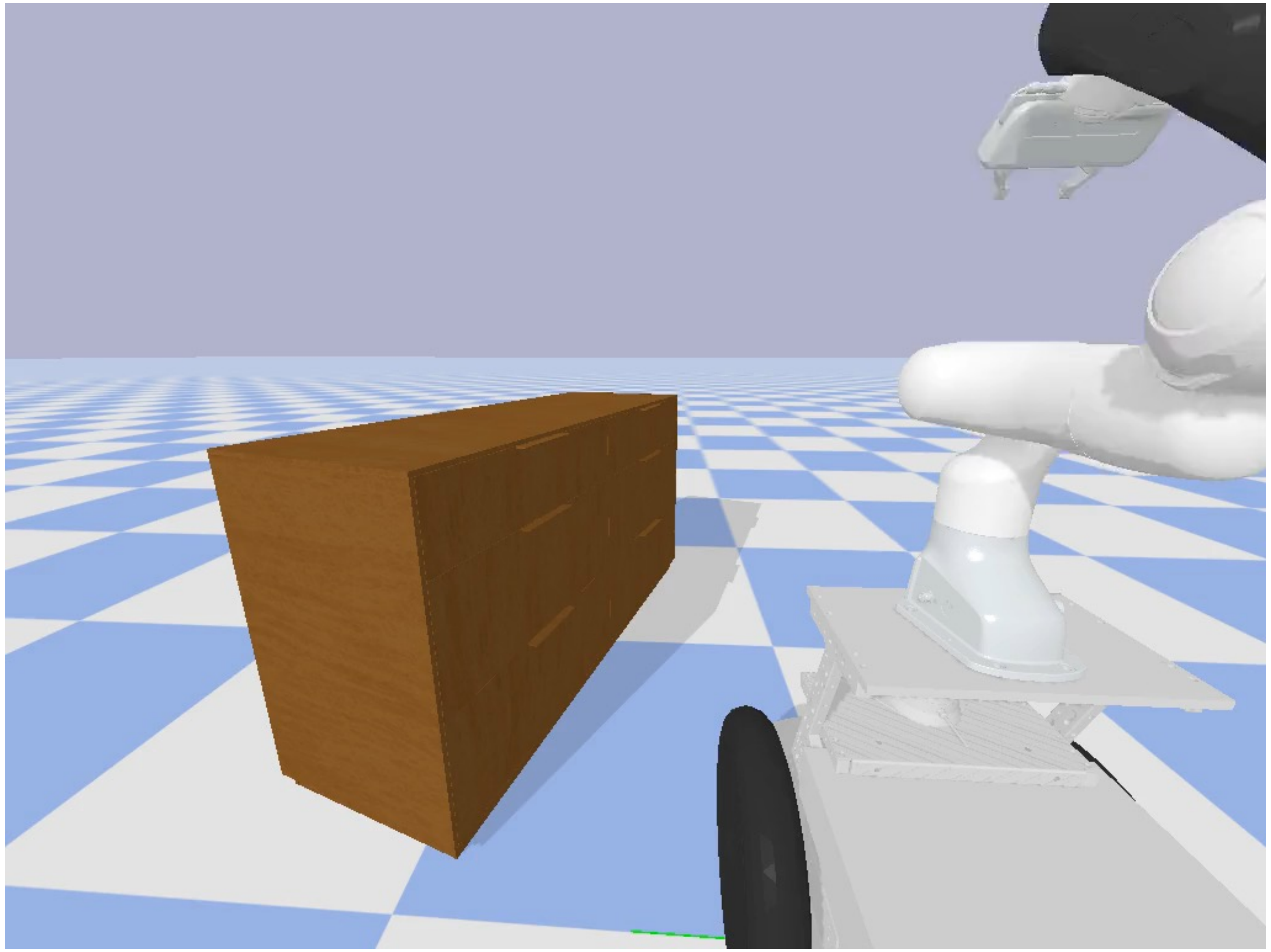
Object Coordinates

The Physics of Virtual Worlds



PyBullet Example

PyBullet Example



Credit: Xiangyun Meng at UW

Physics Simulation

- Dynamical system

State of the virtual world
- Object positions
- Object shapes
- Forces
- Energy

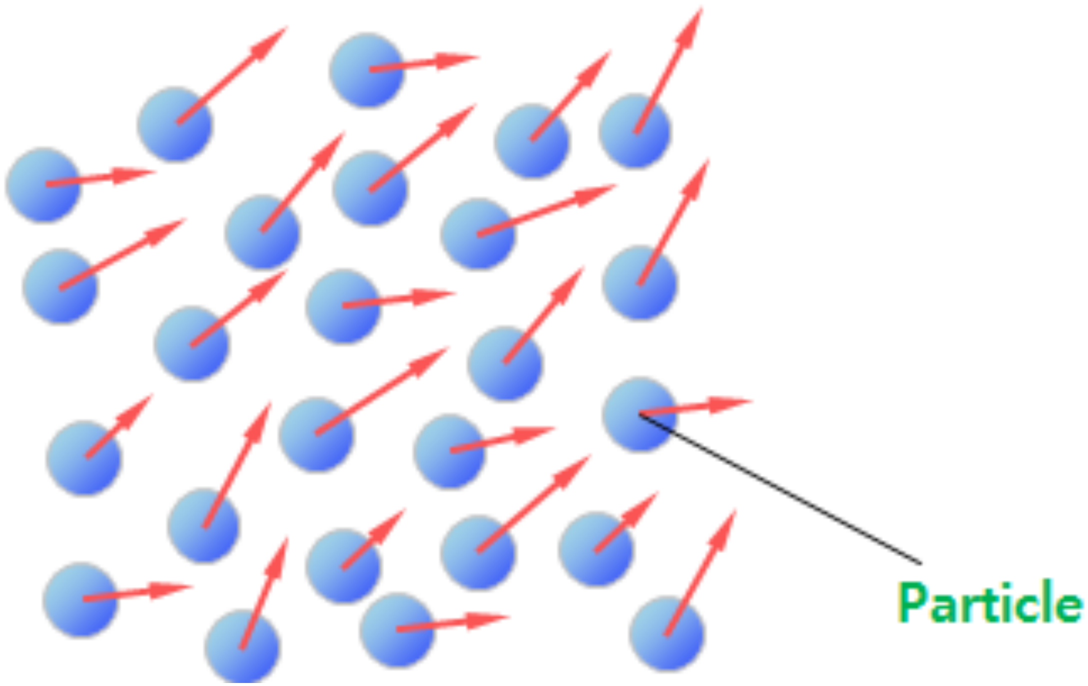
…

$$\mathbf{s}_t \longrightarrow \boxed{\text{Physics Engine}} \longrightarrow \mathbf{s}_{t+1}$$

Pendulum

Particle Dynamics

- Determine the states of particles (e.g., position)



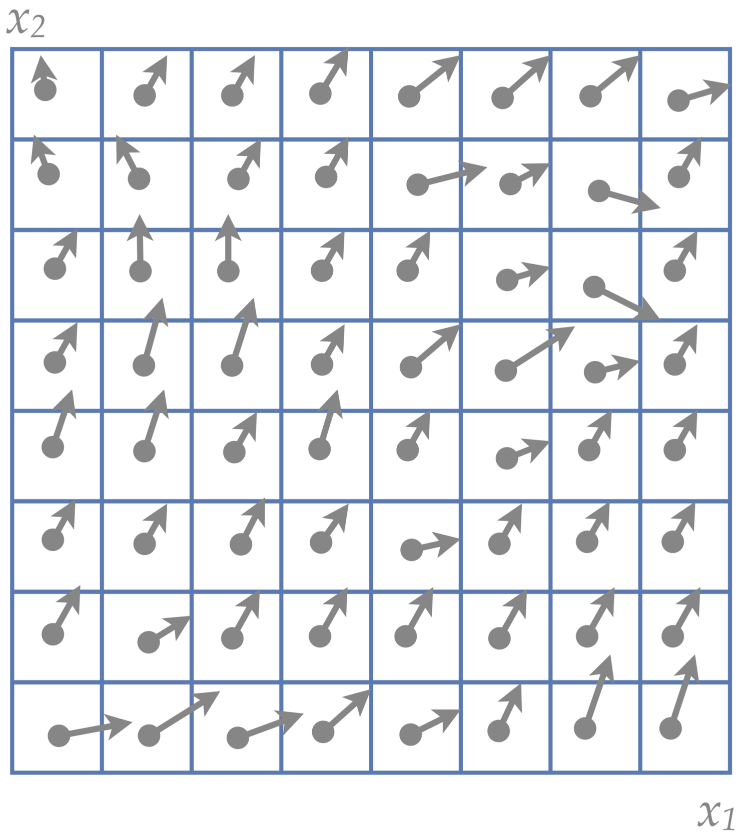
Particle Dynamics

- Determine the position of a mass-less particle
- Given velocity field $\mathbf{v}(\mathbf{x}, t)$
- Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}(\mathbf{x}_p, t)$$

How to calculate $\mathbf{x}_p(t)$



Differential Equations

- A differential equation is an equation that relates one or more functions and their derivatives

$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}(\mathbf{x}_p, t)$$

- Ordinary Differential Equation (ODE)
  - An equation that contains functions of only one independent variable and its derivatives
  - First-order ODE

Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

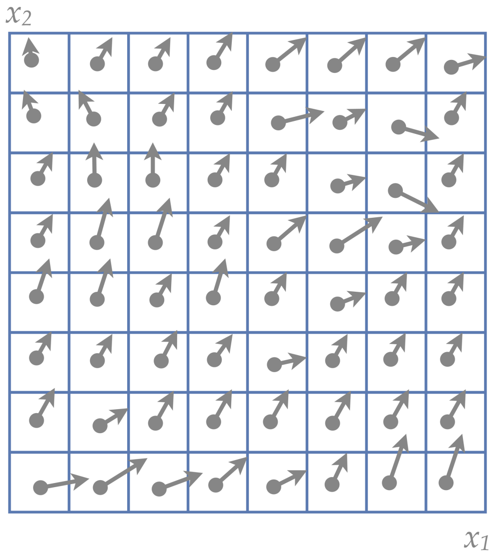
$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}(\mathbf{x}_p, t)$$

- Euler integration

$$\frac{d\mathbf{x}_p(t)}{dt} = \lim_{\epsilon \to 0} \frac{\mathbf{x}_p(t+\epsilon) - \mathbf{x}_p(t)}{\epsilon}$$

$$\frac{d\mathbf{x}_p(t)}{dt} \approx \frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t}$$

$$\frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t} = \mathbf{v}(\mathbf{x}_p, t)$$



Position of the mass-less particle

$$\mathbf{x}_p(t+\Delta t) = \mathbf{x}_p(t) + \Delta t \cdot \mathbf{v}(\mathbf{x}_p, t)$$

Particle Dynamics

- Determine the position of a particle with mass
- Newton's second law

$$\mathbf{f} = m\mathbf{a}$$

Vector sum of all forces applied to each body in a system, newtons (N)

Mass of the body, kg

Vector acceleration of each body with respect to an inertial reference frame, m/sec^2

Acceleration of gravity $g=9.81\ m/sec^2$

Momentum

- The momentum of a body is

$$\mathbf{p}(t) = m\mathbf{v}(t)$$

Mass of the body, kg

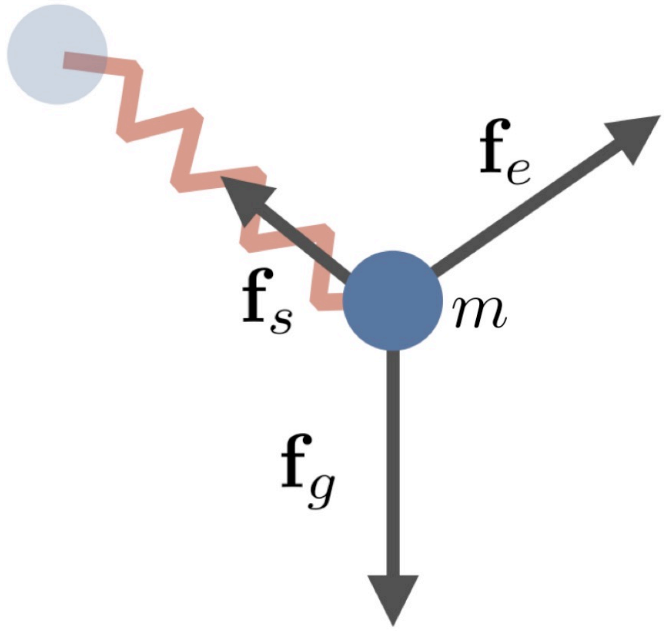
Velocity of the body, m/sec

- Newton's second law

$$\mathbf{f}(t) = \frac{d}{dt}\mathbf{p}(t) = m\frac{d}{dt}\mathbf{v}(t) = m\mathbf{a}(t)$$

Newton's Second Law

- Example



$$ma = \mathbf{f}_s + \mathbf{f}_g + \mathbf{f}_e$$

$\boxed{\text{acceleration}} \longrightarrow \boxed{\text{velocity}} \longrightarrow \boxed{\text{position}}$

Bargteil, A., Shinar T. An introduction to physics-based animation, ACM SIGGRAPH 2018 Courses, 2018

A Particle with Mass

- Initial value problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\frac{d^2\mathbf{x}_p(t)}{dt^2} = \ddot{\mathbf{x}}_p(t) = \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

- First-order equations

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\mathbf{v}_p(0) = \mathbf{v}_0$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}_p(t)$$

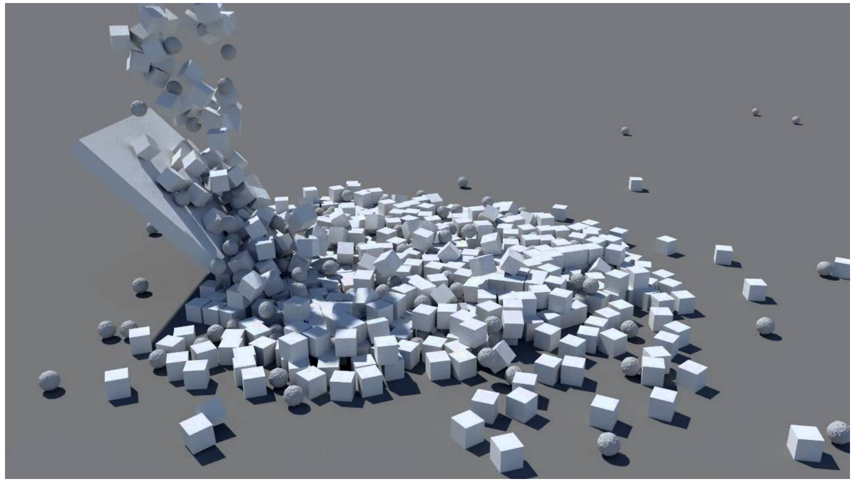
$$\frac{d\mathbf{v}_p(t)}{dt} = \dot{\mathbf{v}}_p(t) = \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

Euler's method

$$\mathbf{v}_p(t + \Delta t) = \mathbf{v}_p(t) + \Delta t \cdot \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

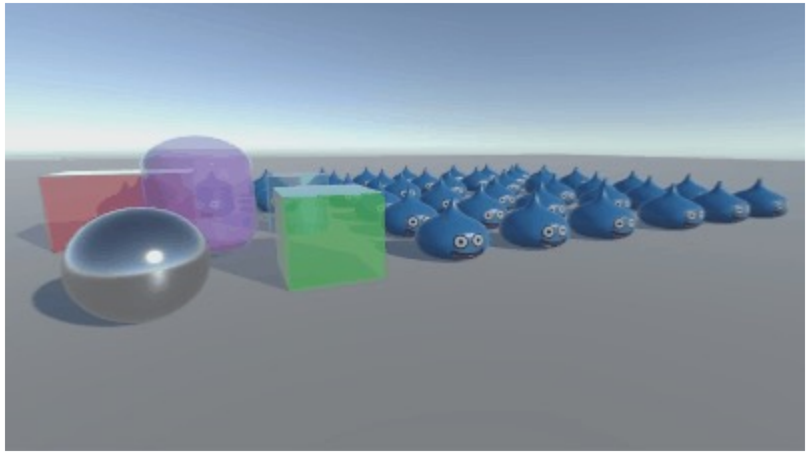
$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + \Delta t \cdot \mathbf{v}_p(t)$$

Materials



Rigid bodies

- No deformation



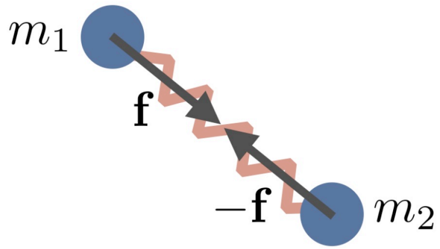
Soft bodies

- Deform elastically and plastically

Fluids

- Air, water, honey, etc.

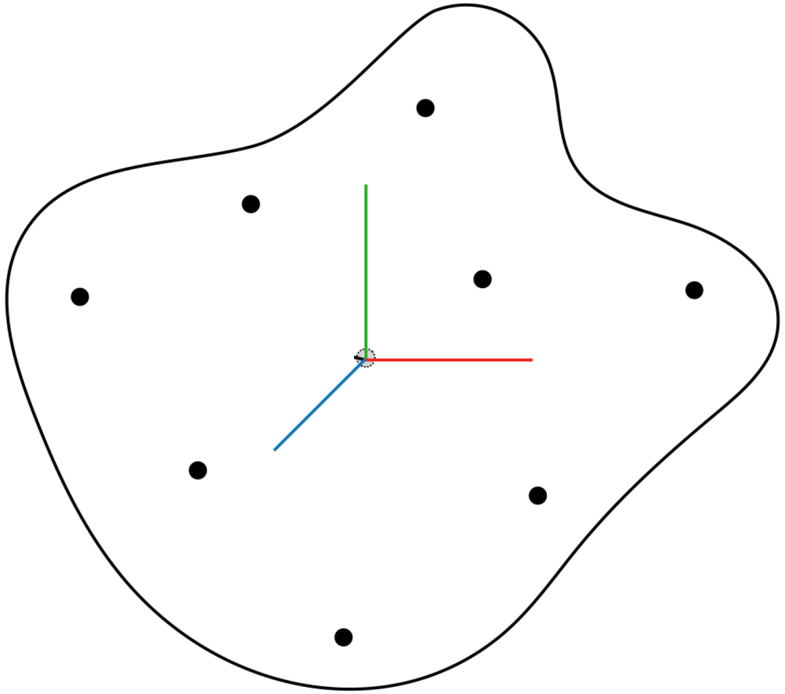
Particles with springs



Rigid Bodies

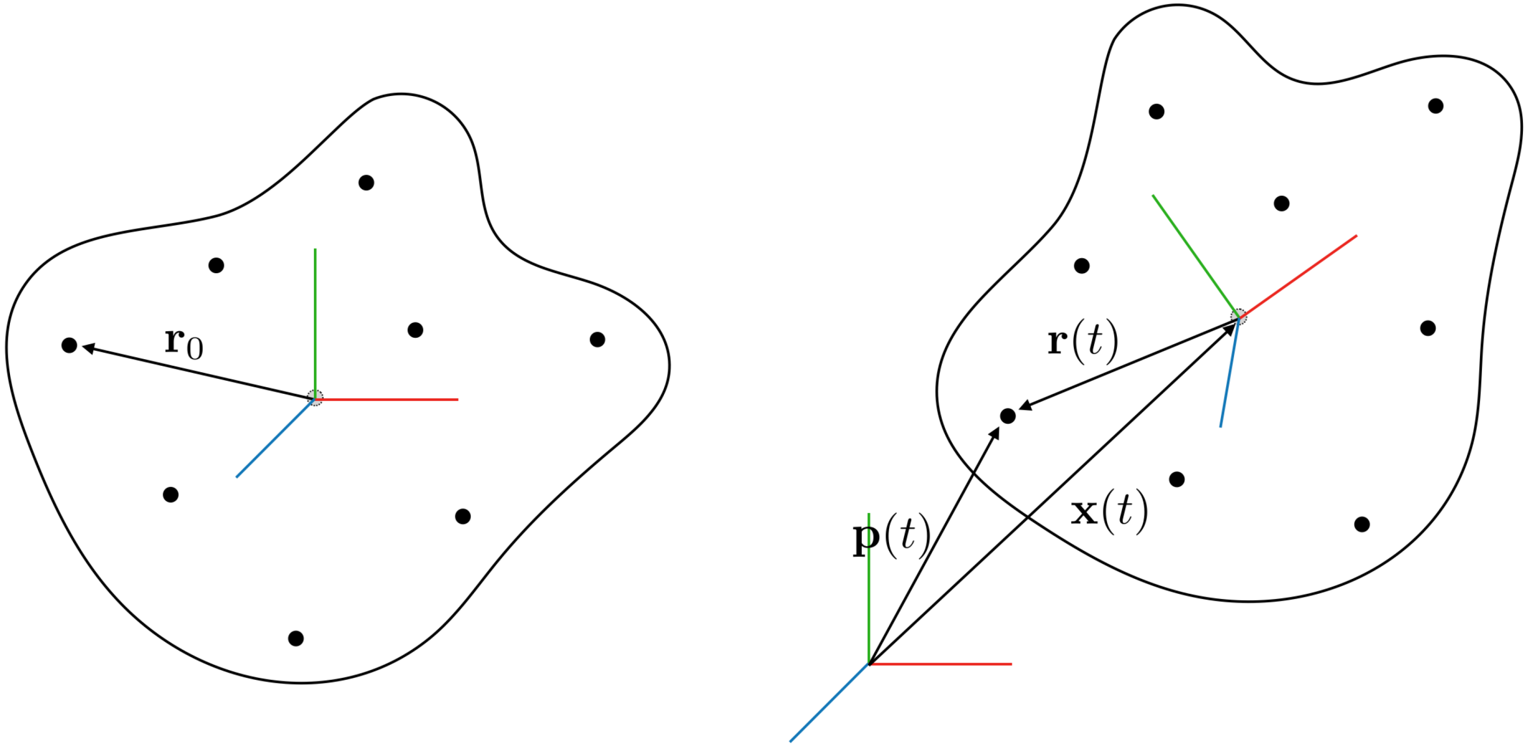
- No deformation
- 6 DOF: 3D translation and 3D rotation
- Particles with very stiff springs
- Center of mass

$$\mathbf{x}_{com} = \frac{\sum_{i=1}^{N} m_i \mathbf{p}_i}{\sum_{i=1}^{N} m_i}$$



Bargteil, A., Shinar T. An introduction to physics-based animation, ACM SIGGRAPH 2018 Courses, 2018

Object Space vs. World Space



(a) *Object space.* **(b)** *World space.*

World position

$$\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_0$$

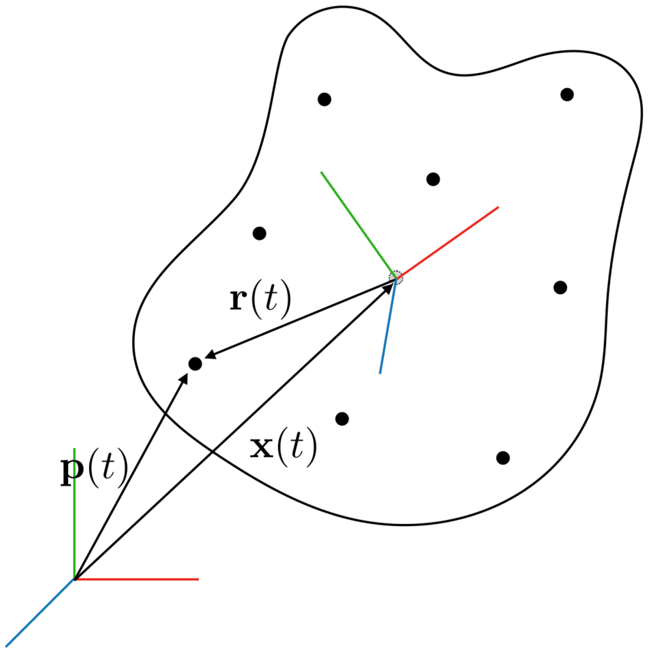
$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{r}(t)$$

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_0$$

Linear Velocity

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_0$$

$$\mathbf{v}(t) = \dot{\mathbf{p}}(t) = \dot{\mathbf{x}}(t) + \dot{\mathbf{R}}(t)\mathbf{r}_0$$



(b) *World space.*

Linear velocity

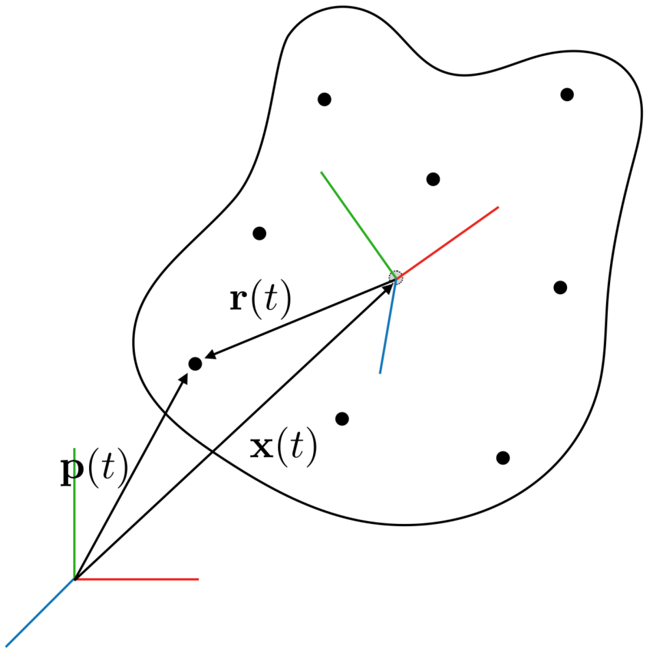
- Motion of the particle due to linear velocity of the body

Instantaneous Rotation

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_0$$

$$\mathbf{v}(t) = \dot{\mathbf{p}}(t) = \dot{\mathbf{x}}(t) + \dot{\mathbf{R}}(t)\mathbf{r}_0$$

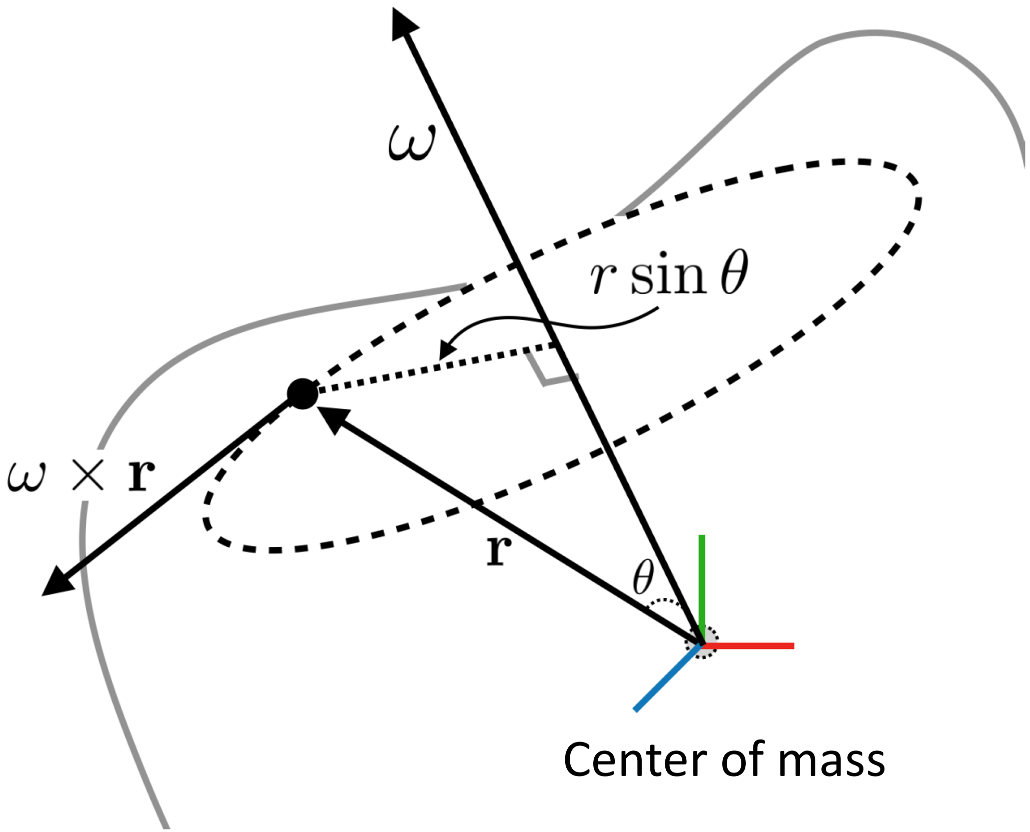
Motion of the particle due to the instantaneous rotation of the body about its center of mass



(b) *World space.*

Angular Velocity ω

Euler's rotation theorem $\dot{\mathbf{R}}(t)\mathbf{r}_0$



ω
- The vector whose direction is the instantaneous axis of rotation
- Length is the rate of rotation in radians per second

$$\dot{\mathbf{R}}(t)\mathbf{r}_0 = \boldsymbol{\omega}(t) \times \mathbf{r}(t)$$

$$\mathbf{v}(t) = \dot{\mathbf{p}}(t) = \dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}(t)$$

$$\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_0 \quad \dot{\mathbf{R}}(t) = \boldsymbol{\omega}(t) \times \mathbf{R}(t)$$

Linear Momentum

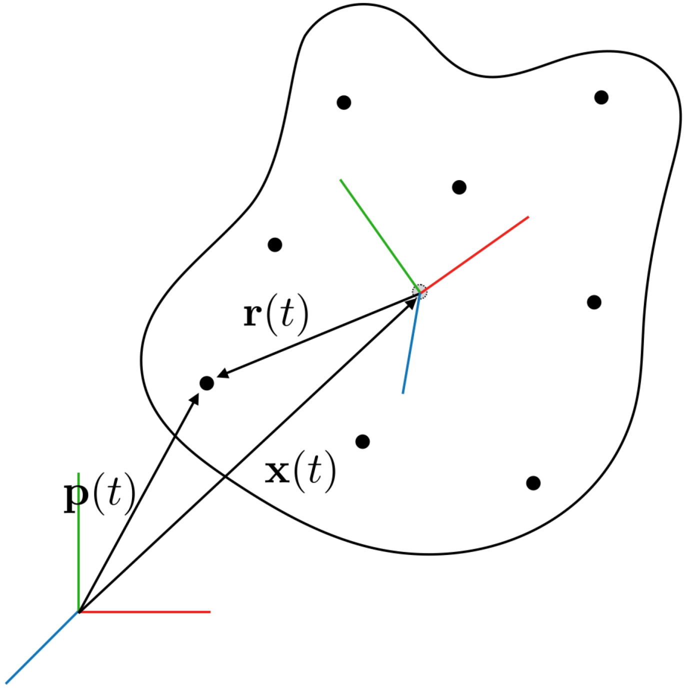
$$\mathbf{P}(t) = \sum_{i=1}^{N} m_i \mathbf{v}_i(t)$$

$$\mathbf{P}(t) = \sum_{i=1}^{N} m_i \left(\dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_i(t)\right)$$

$$= \sum_{i=1}^{N} m_i \dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \left(\sum_{i=1}^{N} m_i \mathbf{r}_i(t)\right)$$

0

$$\mathbf{P}(t) = M\dot{\mathbf{x}}(t) \qquad M = \sum_{i=1}^{N} m_i$$



(b) *World space.*

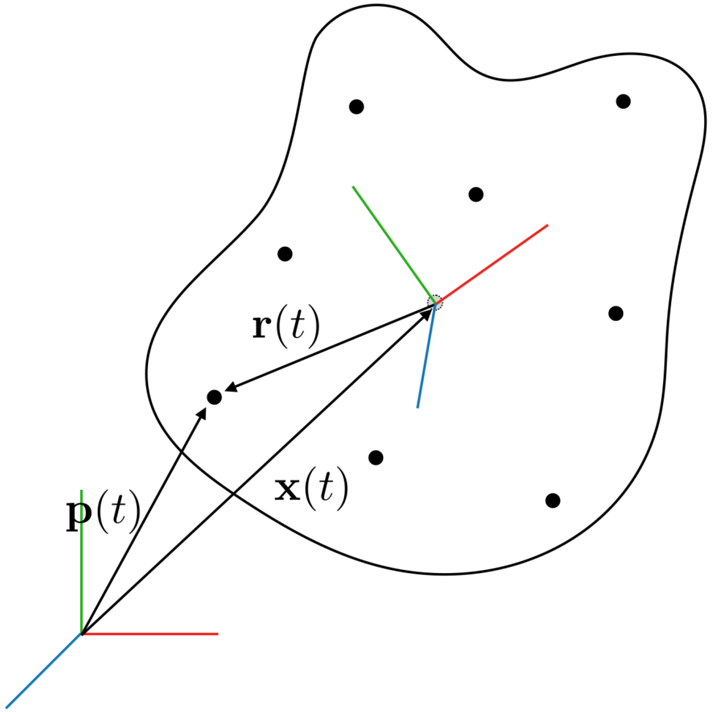
Angular Momentum

$$\mathbf{L}(t) = \sum_{i=1}^{N} \mathbf{r}_i(t) \times m_i \mathbf{v}_i(t)$$

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (\dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_i(t))$$

$$= \underbrace{\sum_{i=1}^{N} m_i \mathbf{r}_i(t)}_{0} \times \dot{\mathbf{x}}(t) + \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times \boldsymbol{\omega}(t) \times \mathbf{r}_i(t)$$

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (\boldsymbol{\omega}(t) \times \mathbf{r}_i(t))$$



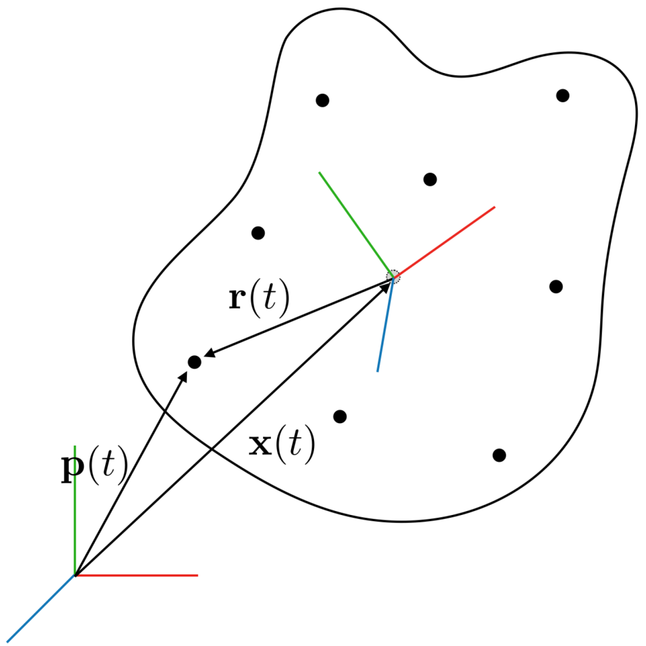
(b) *World space.*

Angular Momentum

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (\boldsymbol{\omega}(t) \times \mathbf{r}_i(t))$$

$$\boldsymbol{\omega} \times \mathbf{r} = -\mathbf{r} \times \boldsymbol{\omega}$$

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (-\mathbf{r}_i(t) \times \boldsymbol{\omega}(t))$$



(b) *World space.*

Cross product matrix $-\mathbf{r}^{\star} = \mathbf{r}^{\star T}$

$$\mathbf{r}^{\star} = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix}$$

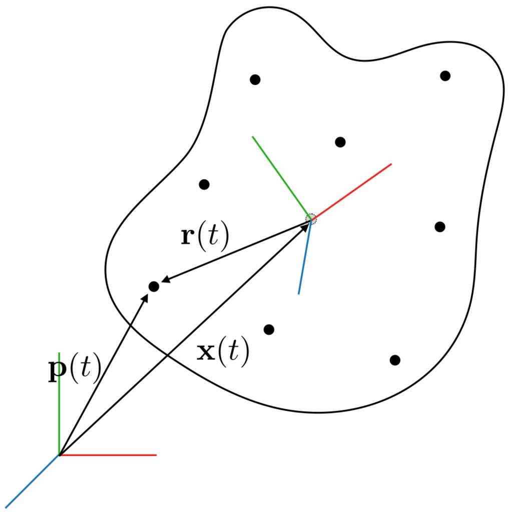
$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^{\star}(t)(\mathbf{r}_i^{\star T}(t)\boldsymbol{\omega}(t))$$

$$= \left(\sum_{i=1}^{N} m_i \mathbf{r}_i^{\star}(t)\mathbf{r}_i^{\star T}(t)\right)\boldsymbol{\omega}(t)$$

Angular Momentum

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)(\mathbf{r}_i^{\star T}(t)\boldsymbol{\omega}(t))$$

$$= \left(\sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)\mathbf{r}_i^{\star T}(t)\right)\boldsymbol{\omega}(t)$$



(b) *World space.*

Inertia tensor

$$\mathbf{I}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)\mathbf{r}_i^{\star T}(t)$$

$$\mathbf{L}(t) = \mathbf{I}(t)\boldsymbol{\omega}(t)$$

$$\begin{aligned}\mathbf{I}(t) &= \sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)\mathbf{r}_i^{\star T}(t)\\ &= \sum_{i=1}^{N} m_i \left(\mathbf{r}_i^T\mathbf{r}_i\boldsymbol{\delta} - \mathbf{r}_i\mathbf{r}_i^T\right)\\ &= \mathbf{R}(t)\sum_{i=1}^{N} m_i \left(\mathbf{r}_{0i}^T\mathbf{r}_{0i}\boldsymbol{\delta} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T\right)\mathbf{R}(t)^T\\ &= \mathbf{R}(t)\mathbf{I}_0\mathbf{R}(t)^T.\end{aligned}$$

$\mathbf{r}^\star\mathbf{r}^{\star T} = \mathbf{r}^T\mathbf{r}\boldsymbol{\delta} - \mathbf{r}\mathbf{r}^T$

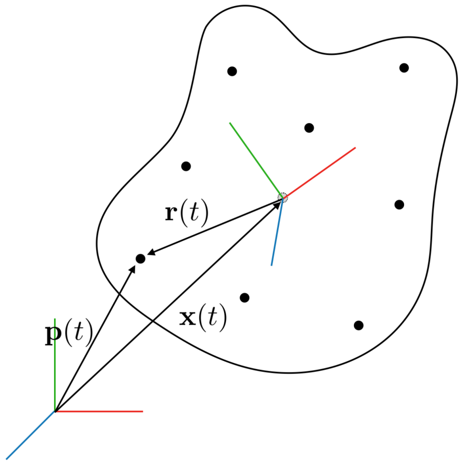
$\boldsymbol{\delta}$ is the 3×3 identity matrix

$\mathbf{r} = \mathbf{R}\mathbf{r}_0$

Force and Torque

Linear momentum $\mathbf{P}(t) = M\dot{\mathbf{x}}(t)$ $\quad M = \sum_{i=1}^{N} m_i$

Angular momentum $\mathbf{L}(t) = \mathbf{I}(t)\boldsymbol{\omega}(t)$



(b) *World space.*

- Newton's second law

$$\frac{d}{dt}\begin{pmatrix}\mathbf{P}(t)\\ \mathbf{L}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{f}(t)\\ \boldsymbol{\tau}(t)\end{pmatrix}$$

Force

Torque

- If a force apply to center of mass

$$\mathbf{a} = \mathbf{f}/M$$

- If a force apply to a point

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$$

Dynamics of Rigid Bodies

$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M} \qquad \mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_0\mathbf{R}(t)^T \qquad \boldsymbol{\omega}(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

$$\frac{d}{dt}\begin{pmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}^{\star}(t)\mathbf{R}(t) \\ \mathbf{f}(t) \\ \boldsymbol{\tau}(t) \end{pmatrix}$$

Linear Velocity

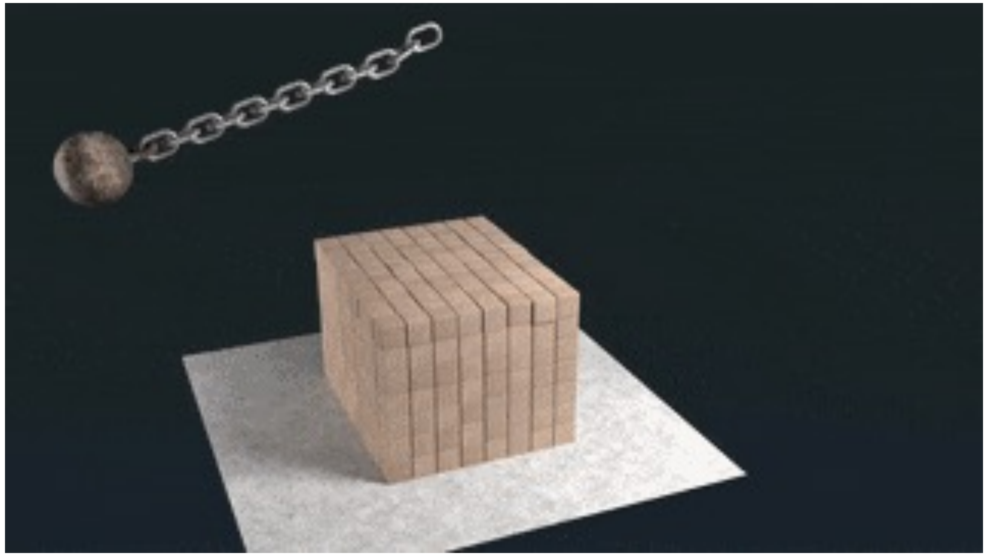
Angular Velocity

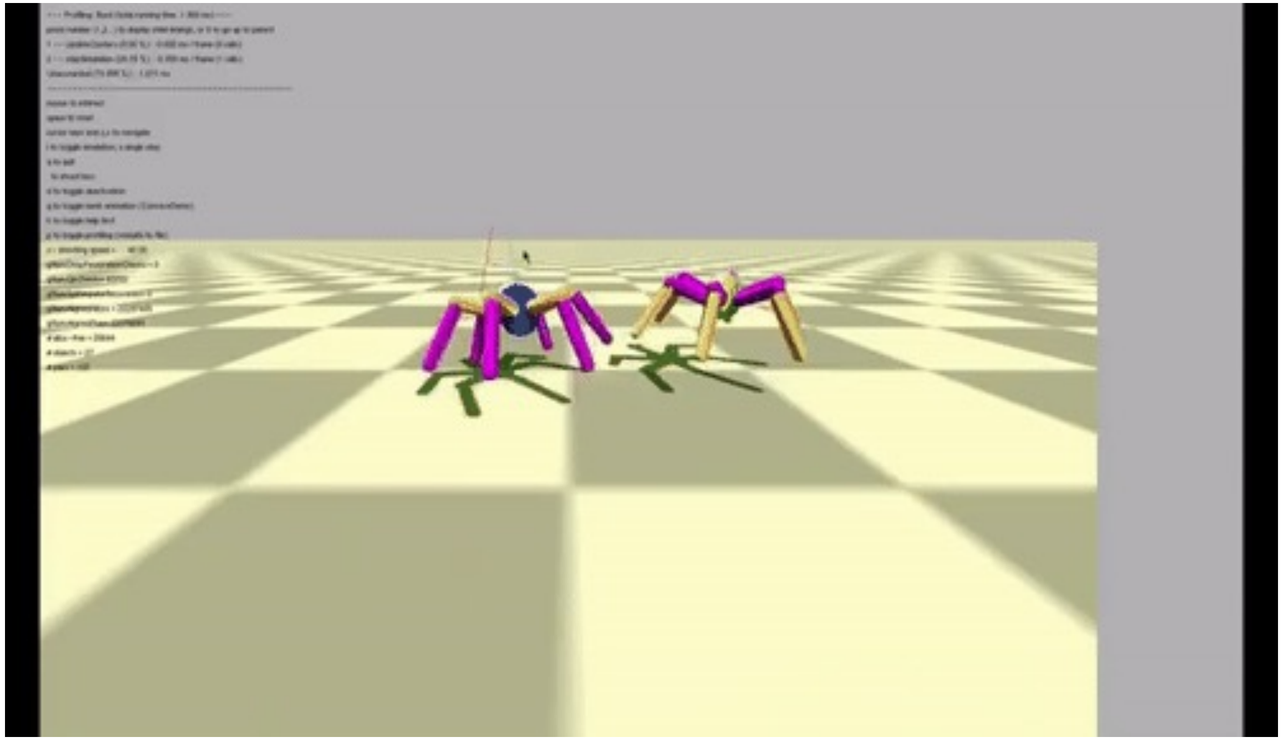
Force

Torque

Rigid Body Simulation Examples

$\mathbf{s}_t \longrightarrow$ Physics Engine $\longrightarrow \mathbf{s}_{t+1}$





https://gfycat.com/

Further Readings

- Section 8.1, 8.3 in Virtual Reality, Steven LaValle
- Bargteil, A., Shinar T. An introduction to physics-based animation, ACM SIGGRAPH 2018 Courses, 2018.